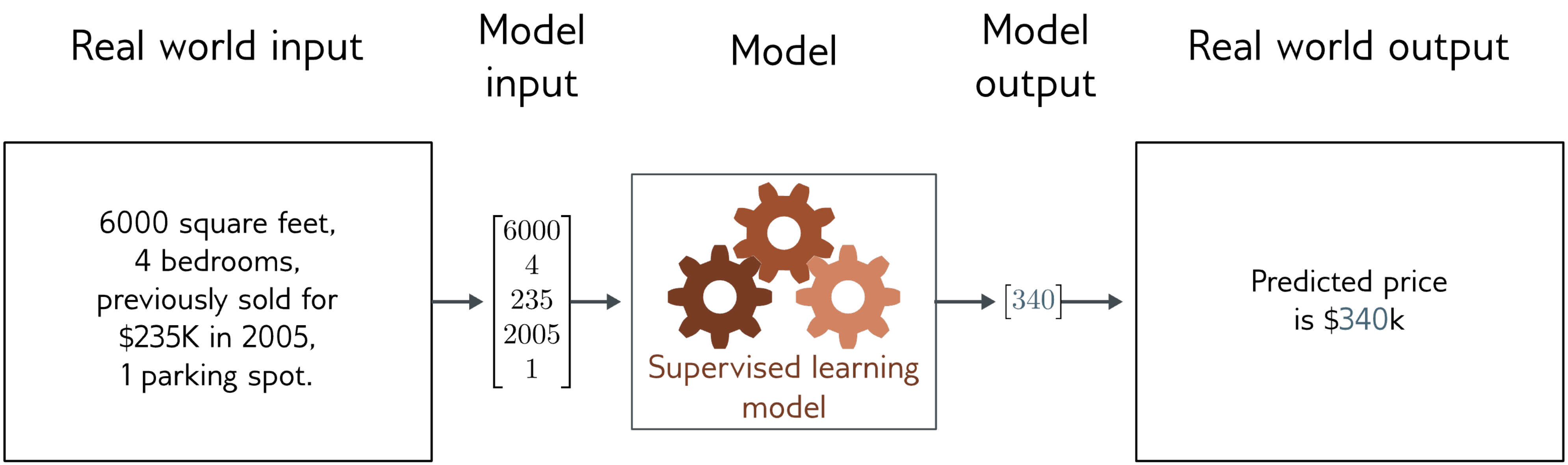
Lecture 03
Shallow Networks

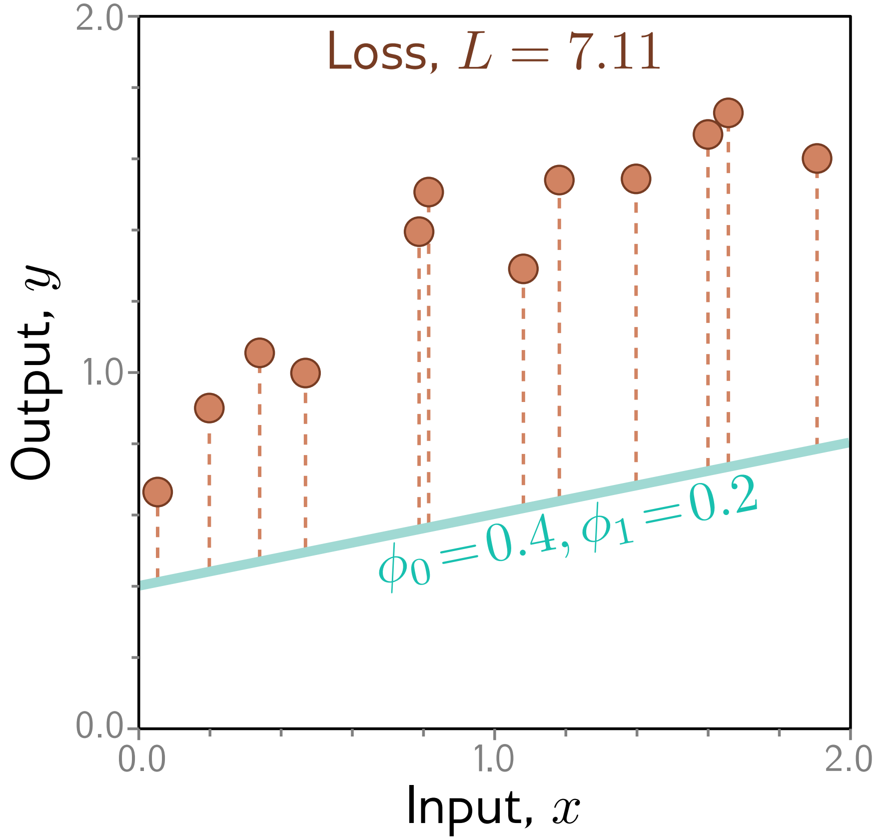
DL4DS – Spring 2025

Recap: Regression



- Univariate regression problem (one output, real value)
- Fully connected network

Recap: 1D Linear regression loss function



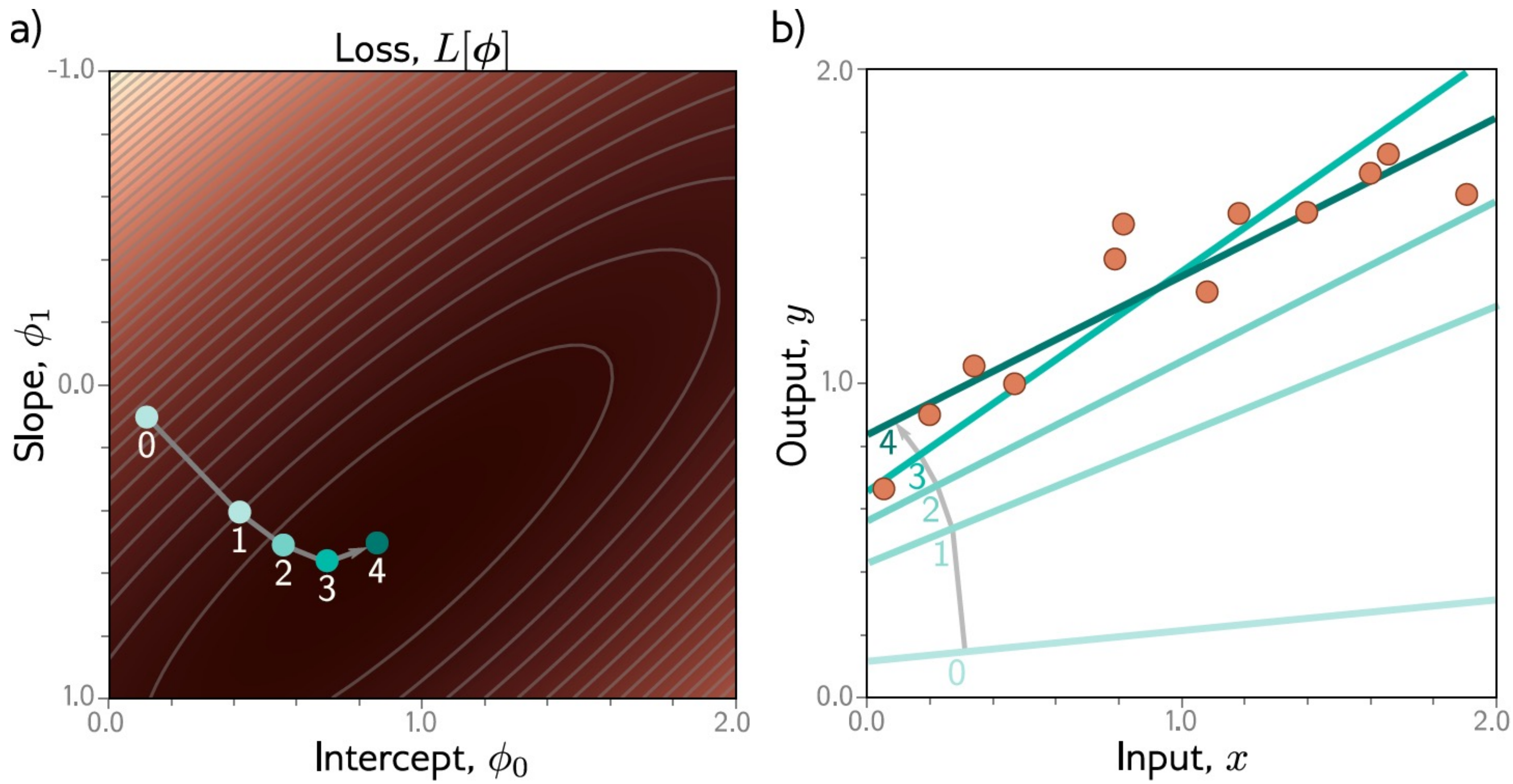
Loss function:

$$L[\boldsymbol{\phi}] = \sum_{i=1}^{I} (\mathrm{f}[x_i, \boldsymbol{\phi}] - y_i)^2$$

$$= \sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i)^2$$

"Least squares loss function"

Recap: 1D Linear regression training



This technique is known as gradient descent

Shallow neural networks

- 1D regression model is obviously limited
  - Want to be able to describe input/output that are not lines
  - Want multiple inputs
  - Want multiple outputs

- Shallow neural networks
  - Flexible enough to describe arbitrarily complex input/output mappings
  - Can have as many inputs as we want
  - Can have as many outputs as we want

This lecture we'll cover…

- Example network, 1 input, 1 output
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

1D Linear Regression

$$
\begin{aligned}
y &= \mathrm{f}[x, \boldsymbol{\phi}] \\
&= \phi_0 + \phi_1 x
\end{aligned}
$$

Example shallow network

$$
\begin{aligned}
y &= \mathrm{f}[x, \boldsymbol{\phi}] \\
&= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11} x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21} x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31} x]
\end{aligned}
$$

Example shallow network

$$
\begin{aligned}
y &= \mathrm{f}[x, \boldsymbol{\phi}] \\
&= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11} x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21} x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31} x]
\end{aligned}
$$

Example shallow network

Activation function

$$
\begin{aligned}
y &= \mathrm{f}[x, \boldsymbol{\phi}] \\
&= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x]
\end{aligned}
$$

Example shallow network

Activation function

$$
\begin{aligned}
y &= \mathrm{f}[x, \boldsymbol{\phi}] \\
&= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11} x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21} x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31} x]
\end{aligned}
$$

$$
\mathrm{a}[z] = \mathrm{ReLU}[z] = \begin{cases} 0 & z < 0 \\ z & z \geq 0 \end{cases}.
$$

Rectified Linear Unit

(one type of activation function)

Example shallow network

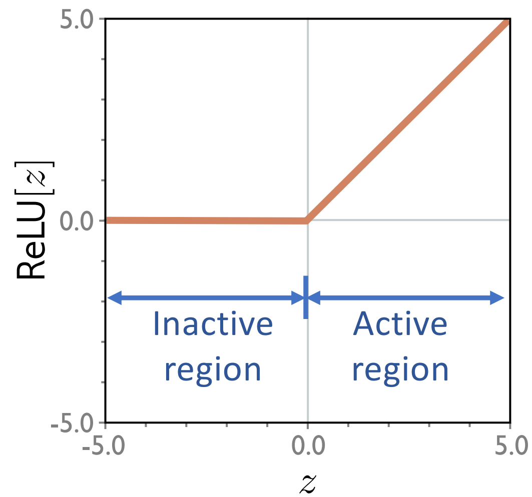
Activation function

$$
\begin{aligned}
y &= \mathrm{f}[x, \boldsymbol{\phi}] \\
&= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x]
\end{aligned}
$$

$$
\mathrm{a}[z] = \mathrm{ReLU}[z] = \begin{cases} 0 & z < 0 \\ z & z \geq 0 \end{cases}.
$$

Rectified Linear Unit

(particular kind of activation function)



Example shallow network

$$y = \mathrm{f}[x, \boldsymbol{\phi}]$$
$$= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x]$$

This model has 10 parameters:

$$\boldsymbol{\phi} = \{\phi_0, \phi_1, \phi_2, \phi_3, \theta_{10}, \theta_{11}, \theta_{20}, \theta_{21}, \theta_{30}, \theta_{31}\}$$

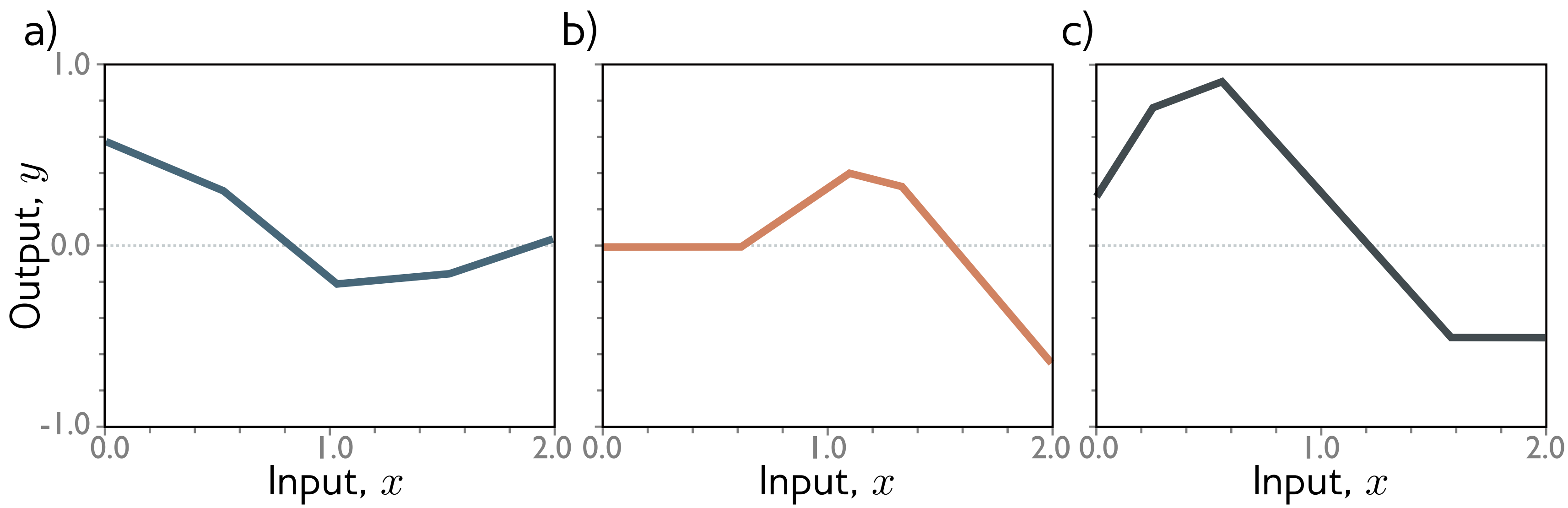
- Represents a family of functions
- Parameters determine a particular function
- Given the parameters, we can perform inference (evaluate the equation)
- Given training dataset $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^{I}$
- Define loss function $L[\boldsymbol{\phi}]$ (least squares)
- Change parameters to minimize loss function

Example shallow network

$$y = \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x].$$

Example shallow network

$$y = \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11} x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21} x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31} x].$$



Piecewise linear functions with three joints

Hidden units

$$y = \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x].$$

Break down into two parts:

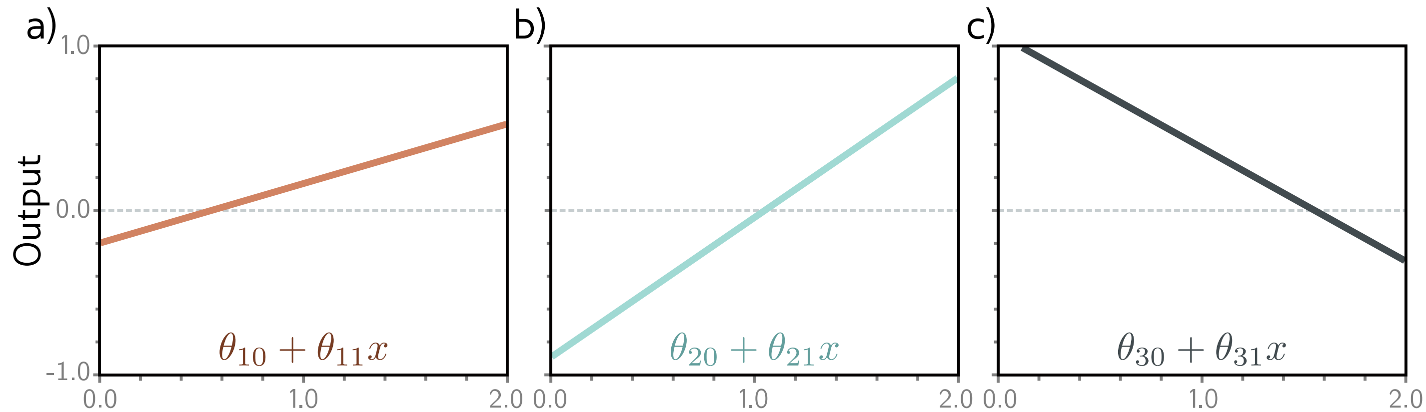
$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

where:

$$\text{Hidden units} \left\{ \begin{array}{l} h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x] \\ h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x] \\ h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x] \end{array} \right.$$

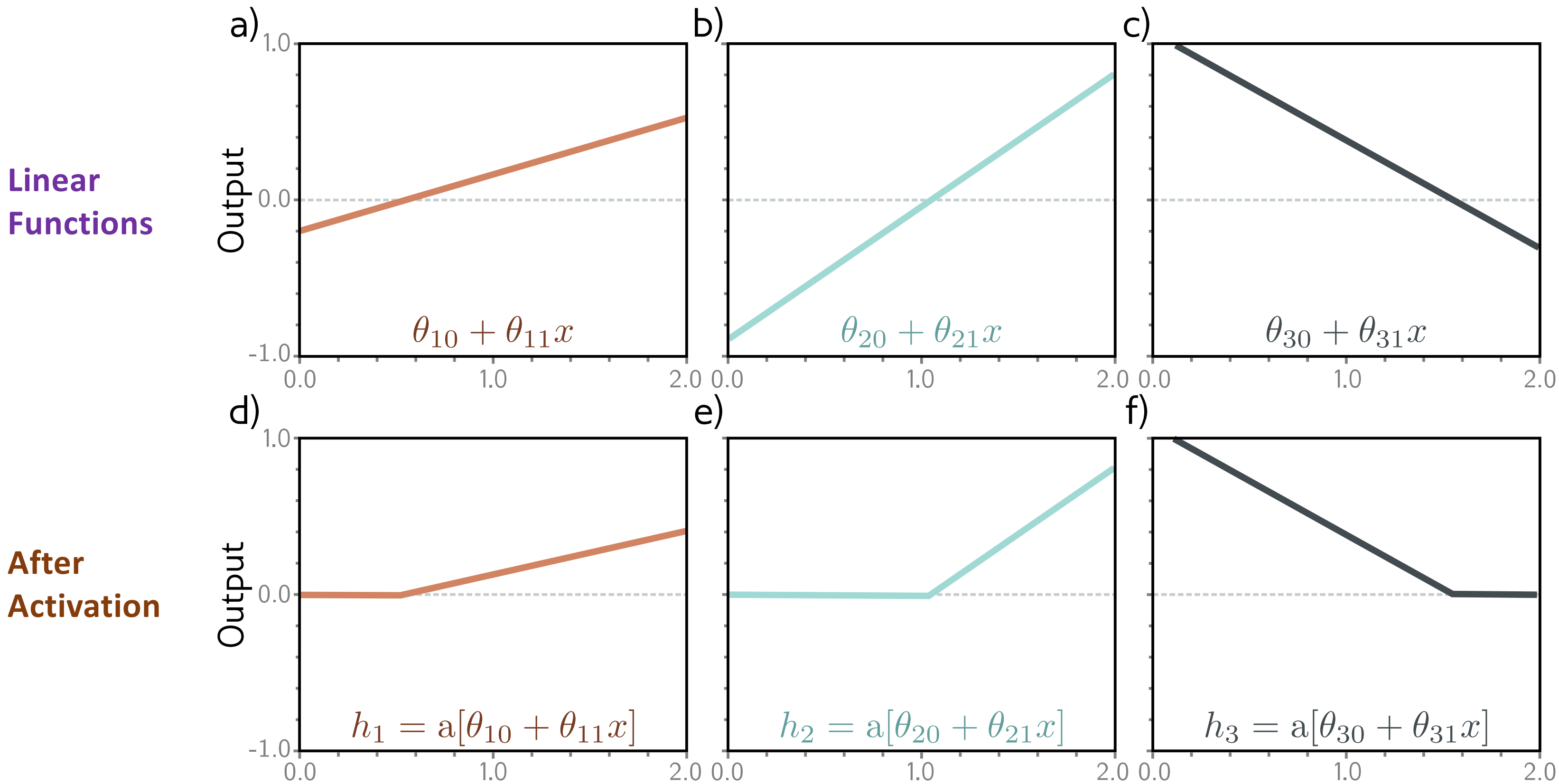
1. compute three linear functions

Linear Functions

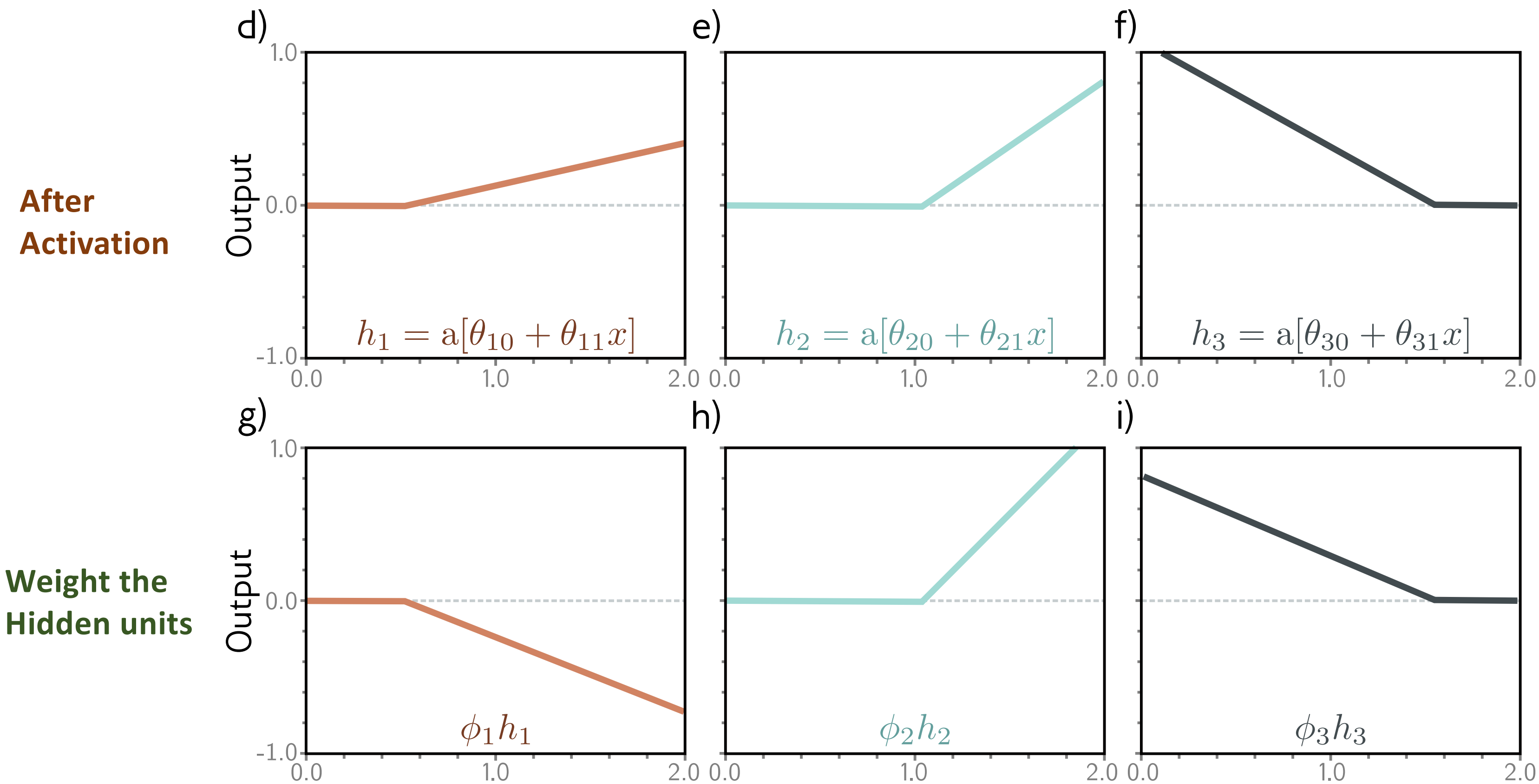


2. Pass through ReLU functions (creates hidden units)

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x],$$



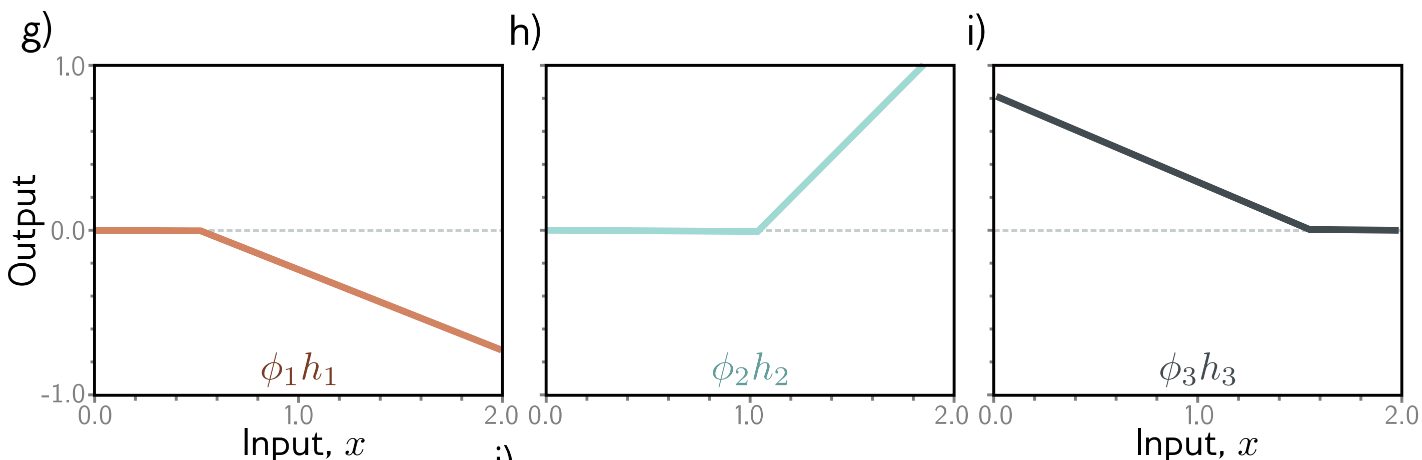
2. Weight the hidden units



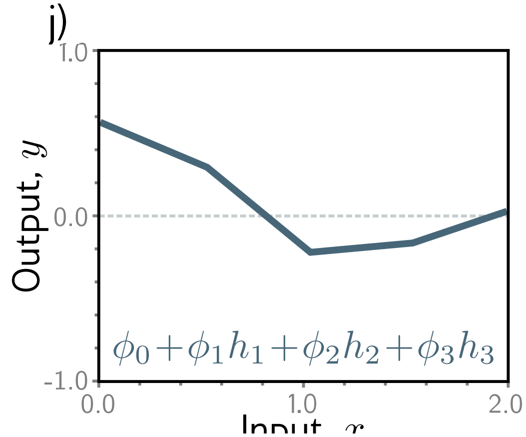
4. Sum the weighted hidden units to create output

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

Weight the hidden units

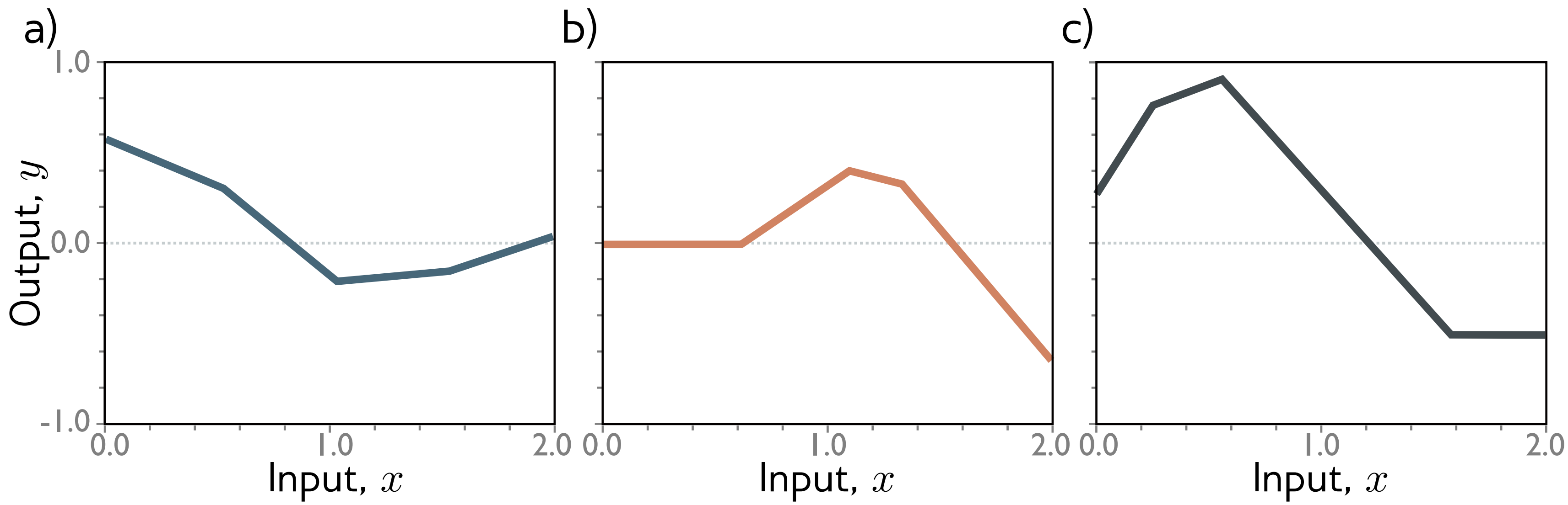


Sum the weighted hidden units



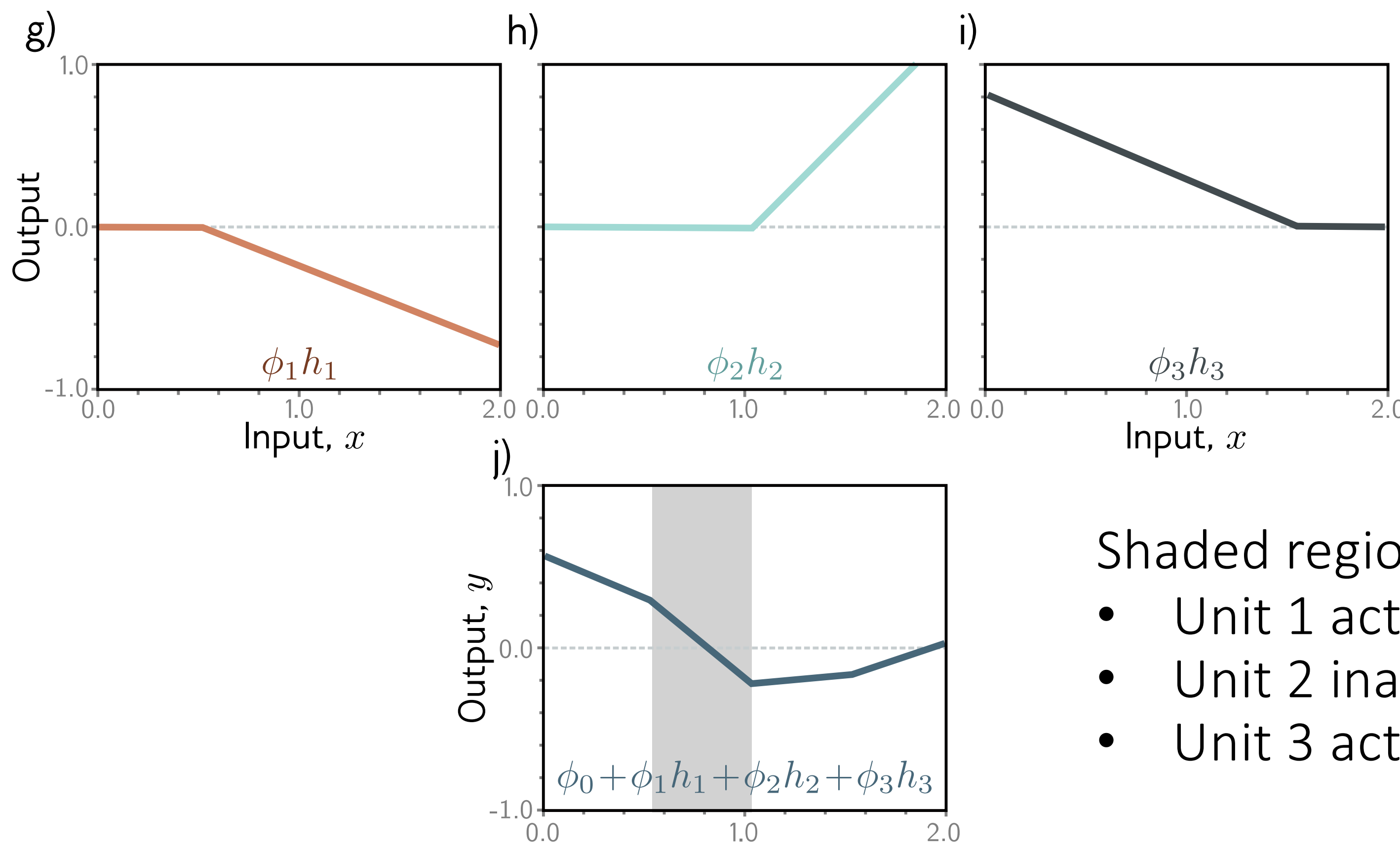
Example: 3 different shallow networks

$$y = \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x].$$



Example shallow network = piecewise linear functions
1 "joint" per ReLU function

Activation pattern = which hidden units are activated?



Shaded region:

- Unit 1 active
- Unit 2 inactive
- Unit 3 active

Interactive Figure 3.3a: 1D Shallow Network (ReLU)

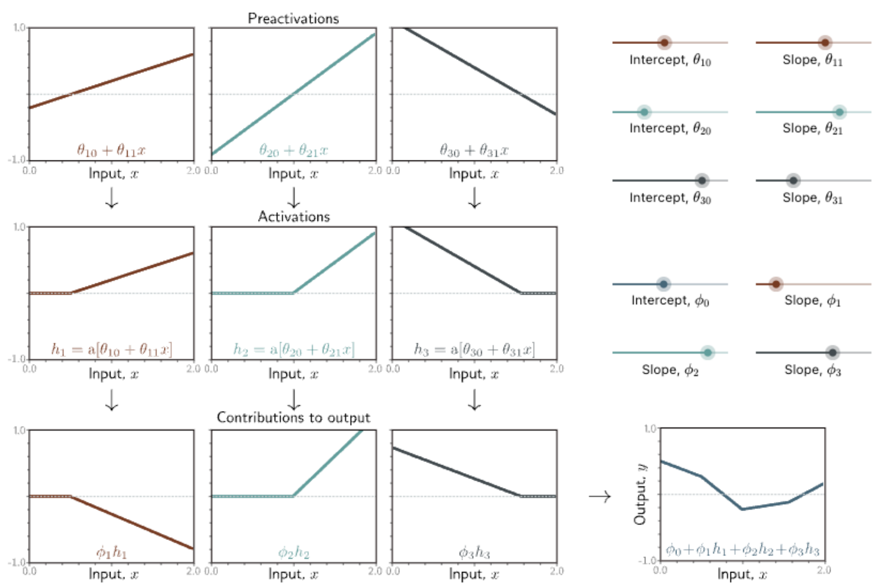


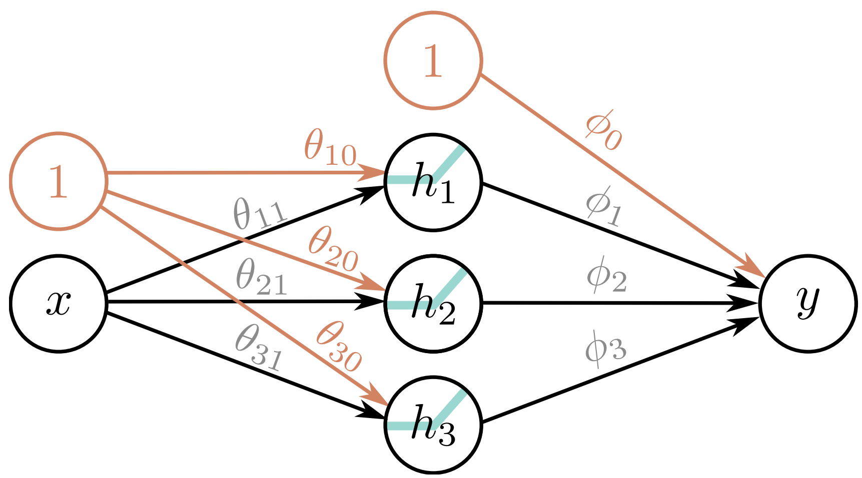
Figure 3.3 Computation for function in figure 3.2a. (Top row) The input x is passed through three linear functions, each with a different y-intercept $\theta_{\bullet 0}$ and slope $\theta_{\bullet 1}$. (Center row) Each line is passed through the ReLU activation function. (Bottom row) The three resulting functions are then weighted (scaled) by ϕ_1, ϕ_2, and ϕ_3, respectively. (Bottom right) Finally, the weighted functions are summed, and an offset ϕ_0 that controls the height is added.

Move the sliders to modify the parameters of the shallow network.

https://udlbook.github.io/udlfigures/

Depicting neural networks

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x] \qquad y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$



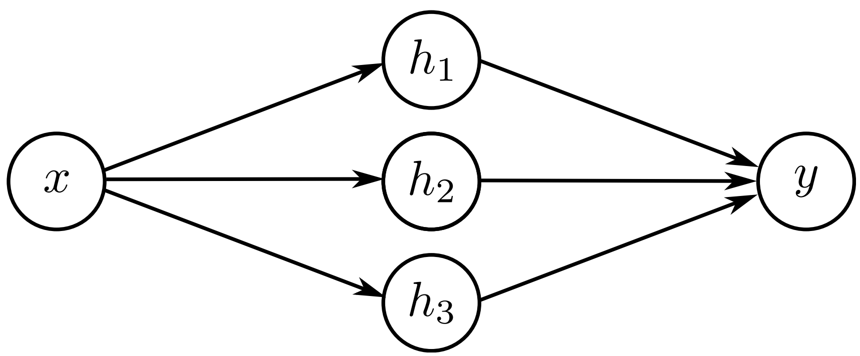
Each parameter multiplies its source and adds to its target

Depicting neural networks

Usually don't show the bias terms

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$



Shallow neural networks

- Example network, 1 input, 1 output
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

With 3 hidden units:

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

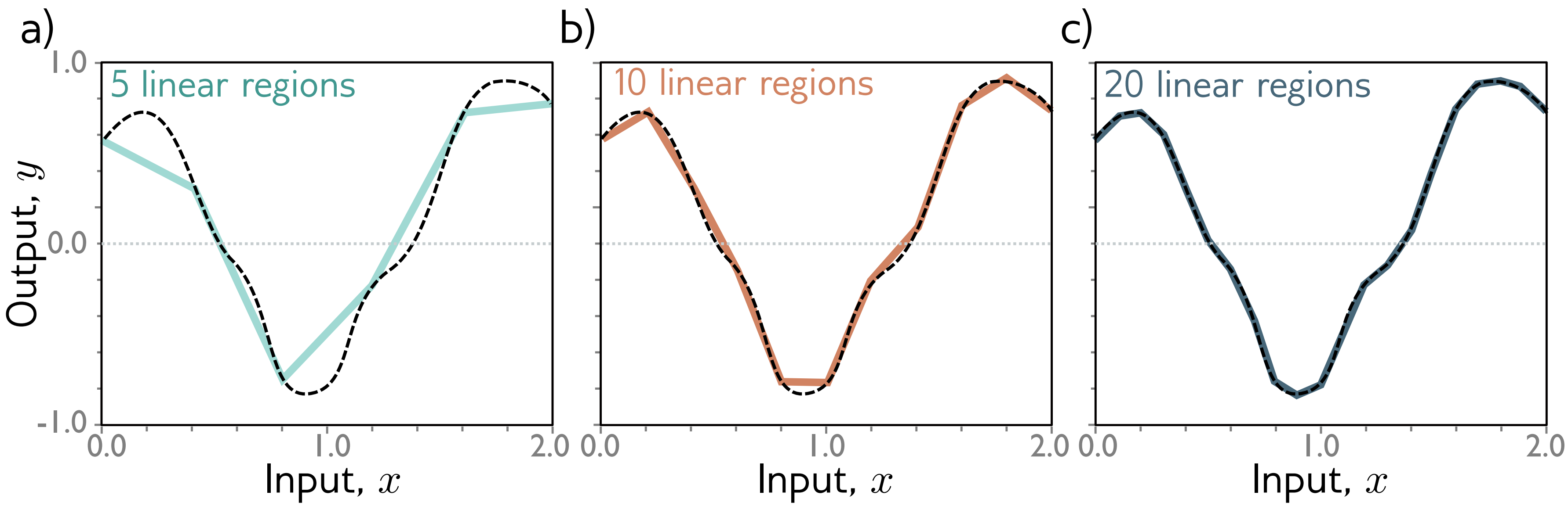
With D hidden units:

$$h_d = \mathrm{a}[\theta_{d0} + \theta_{d1}x]$$

$$y = \phi_0 + \sum_{d=1}^{D} \phi_d h_d$$

With enough hidden units…

… we can describe any 1D function to arbitrary accuracy



Universal approximation theorem

"a formal proof that, with enough hidden units, a shallow neural network can describe any continuous function in $\mathbb{R}^D$ to arbitrary precision"

Shallow neural networks

- Example network, 1 input, 1 ouput
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

Two outputs

- 1 input, 4 hidden units, 2 outputs

$$
\begin{aligned}
h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x] \\
h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x] \\
h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x] \\
h_4 &= \mathrm{a}[\theta_{40} + \theta_{41}x]
\end{aligned}
$$

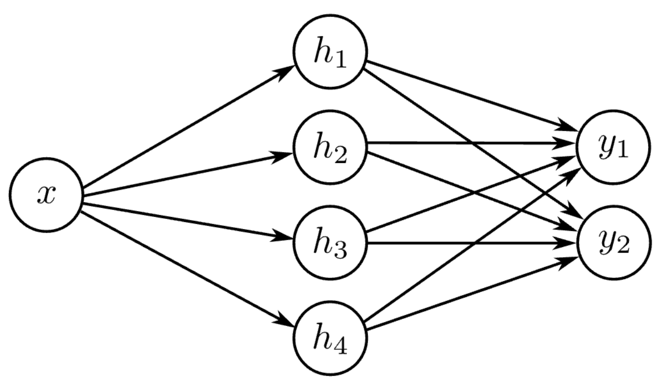
$$
\begin{aligned}
y_1 &= \phi_{10} + \phi_{11}h_1 + \phi_{12}h_2 + \phi_{13}h_3 + \phi_{14}h_4 \\
y_2 &= \phi_{20} + \phi_{21}h_1 + \phi_{22}h_2 + \phi_{23}h_3 + \phi_{24}h_4
\end{aligned}
$$

Two outputs

- 1 input, 4 hidden units, 2 outputs

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$
$$h_4 = \mathrm{a}[\theta_{40} + \theta_{41}x]$$

$$y_1 = \phi_{10} + \phi_{11}h_1 + \phi_{12}h_2 + \phi_{13}h_3 + \phi_{14}h_4$$
$$y_2 = \phi_{20} + \phi_{21}h_1 + \phi_{22}h_2 + \phi_{23}h_3 + \phi_{24}h_4$$



Two outputs

- 1 input, 4 hidden units, 2 outputs

$$
\begin{aligned}
h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x] \\
h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x] \\
h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x] \\
h_4 &= \mathrm{a}[\theta_{40} + \theta_{41}x]
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= \phi_{10} + \phi_{11}h_1 + \phi_{12}h_2 + \phi_{13}h_3 + \phi_{14}h_4 \\
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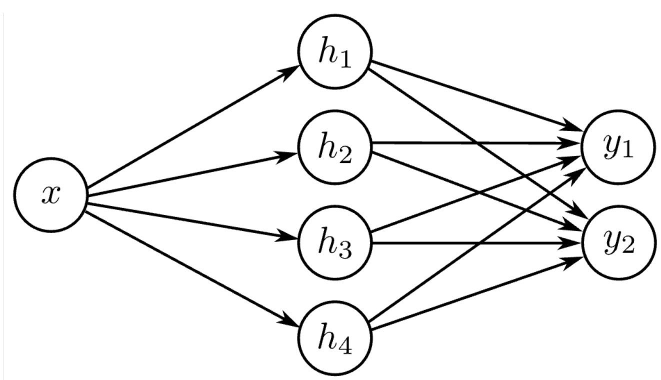
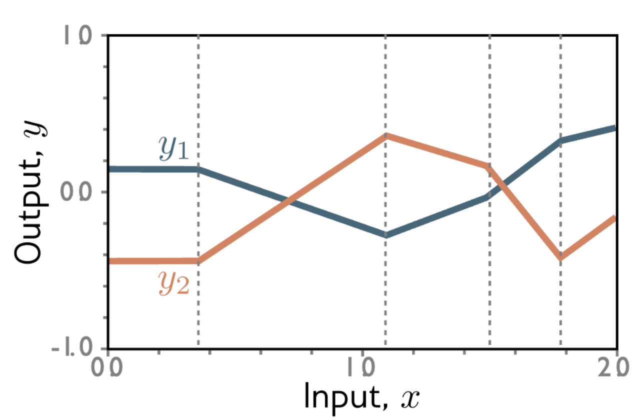
\end{aligned}
$$





Shallow neural networks

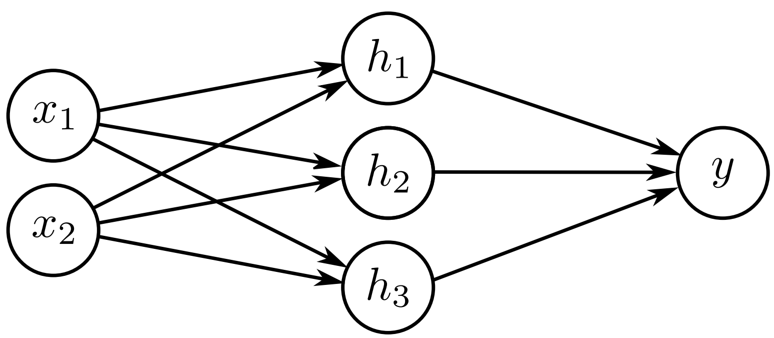
- Example network, 1 input, 1 ouput
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

Two inputs

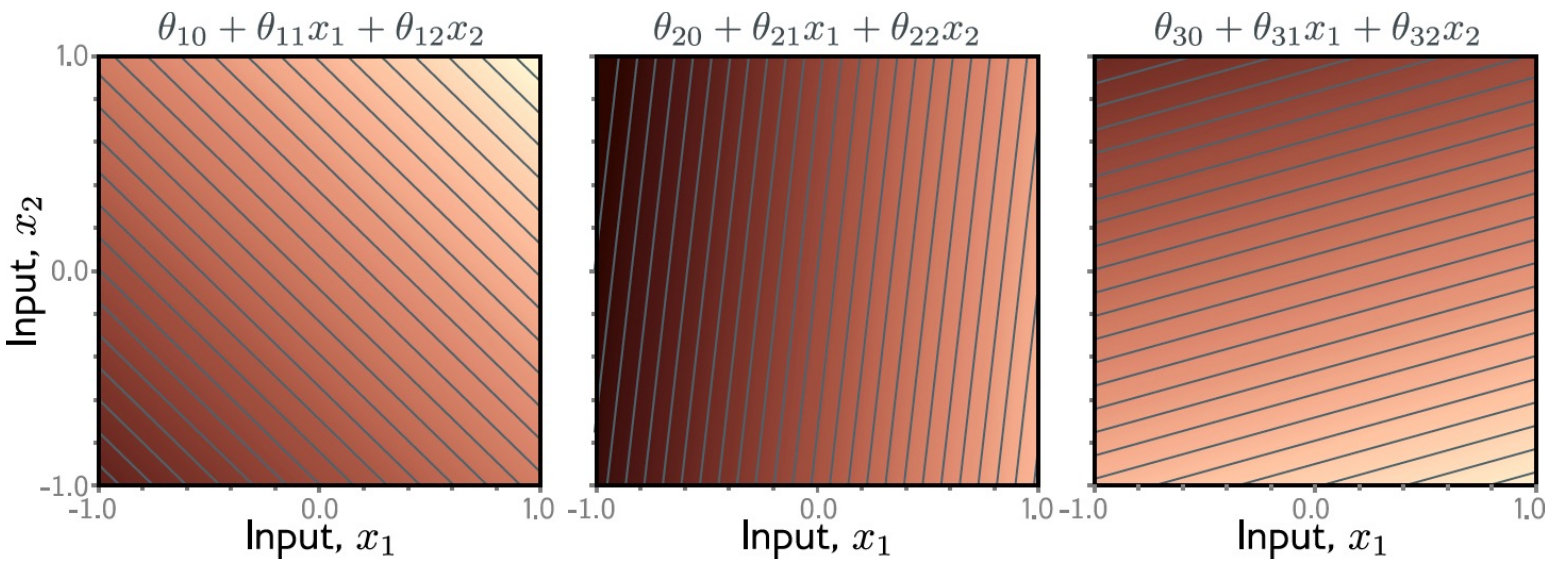
- 2 inputs, 3 hidden units, 1 output

$$
\begin{aligned}
h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x_1 + \theta_{12}x_2] \\
h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x_1 + \theta_{22}x_2] \\
h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x_1 + \theta_{32}x_2]
\end{aligned}
$$

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$



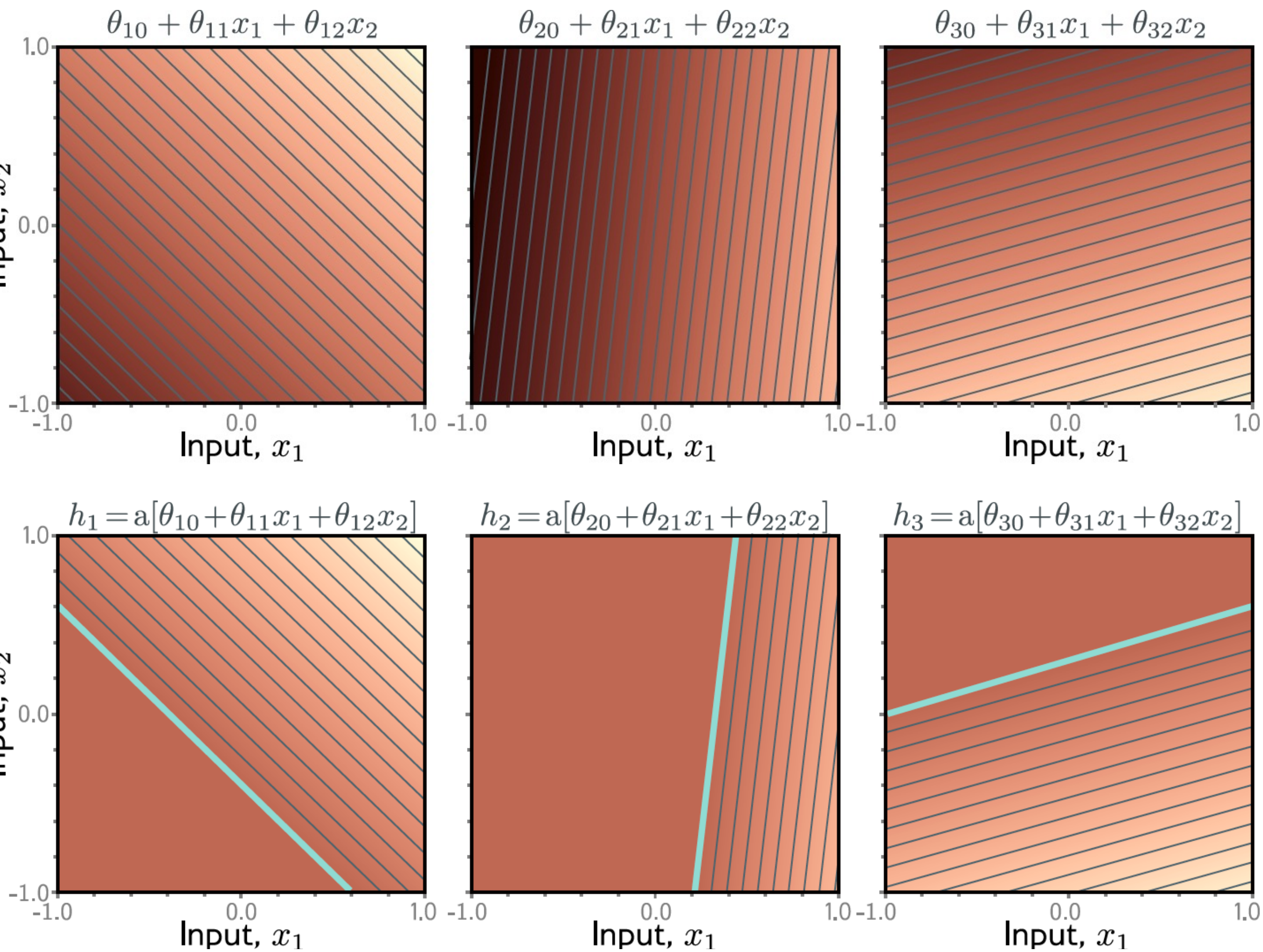
Linear Functions



See Interactive Figure 3.8a https://udlbook.github.io/udlfigures/

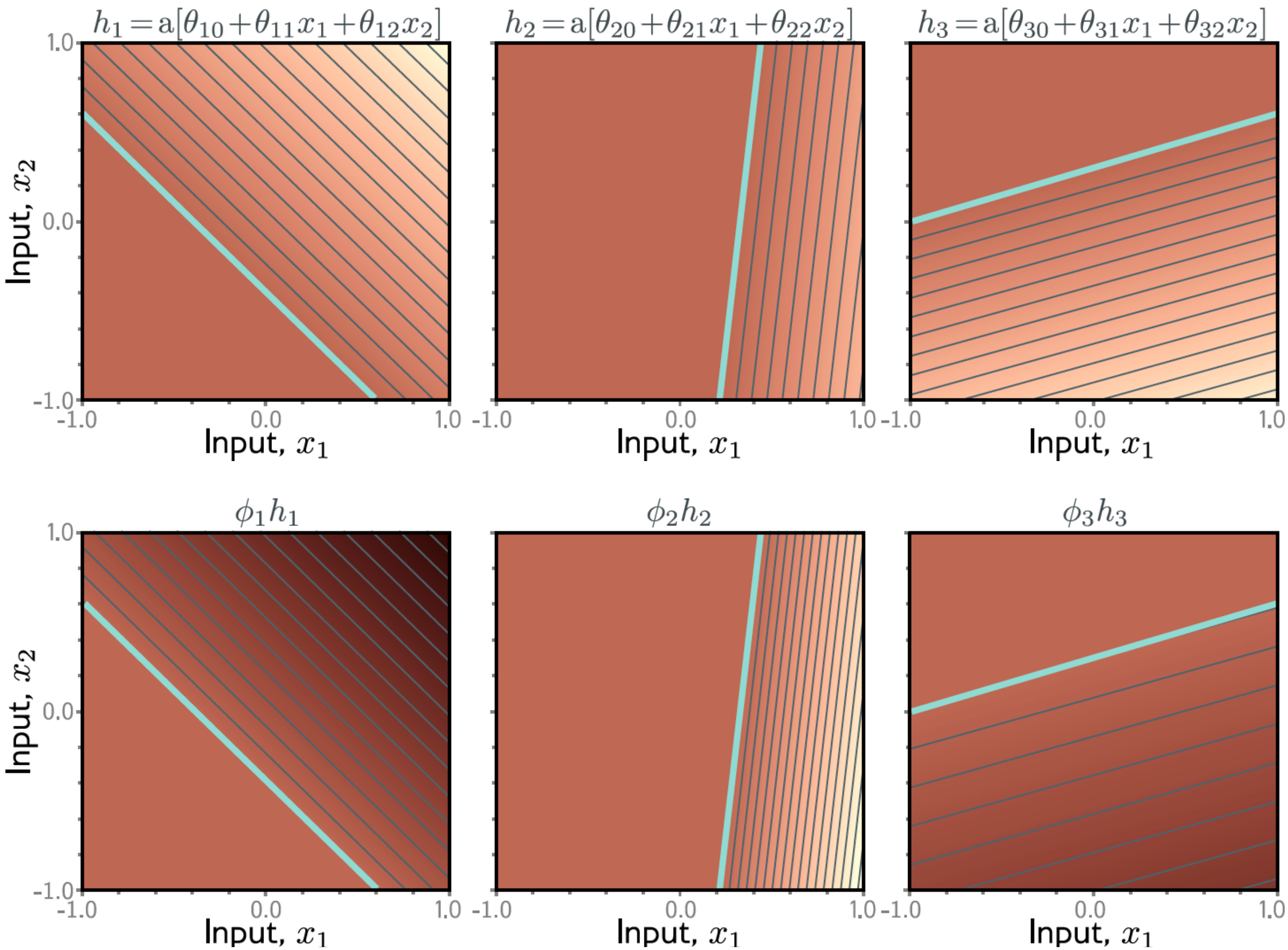
Linear Functions

After Activation

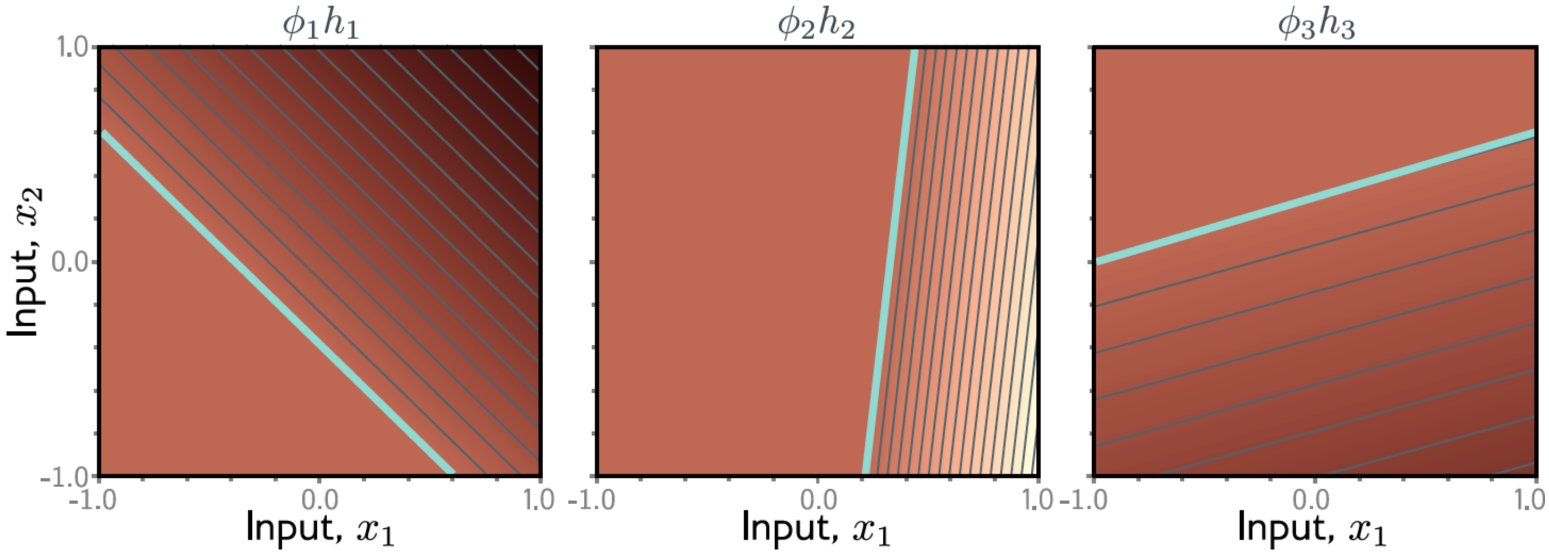


After Activation

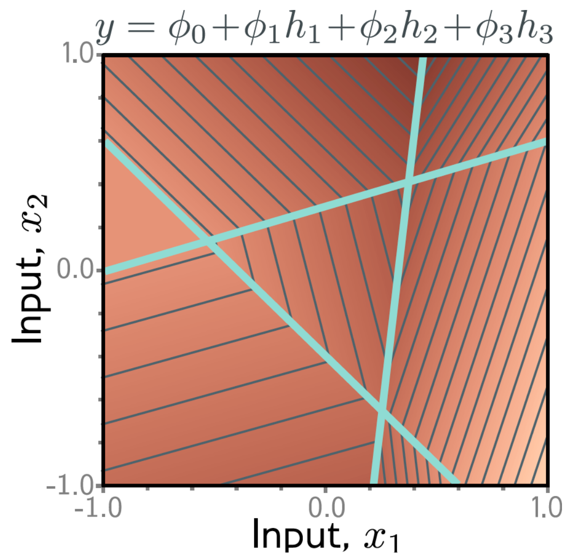
Weight the Hidden units

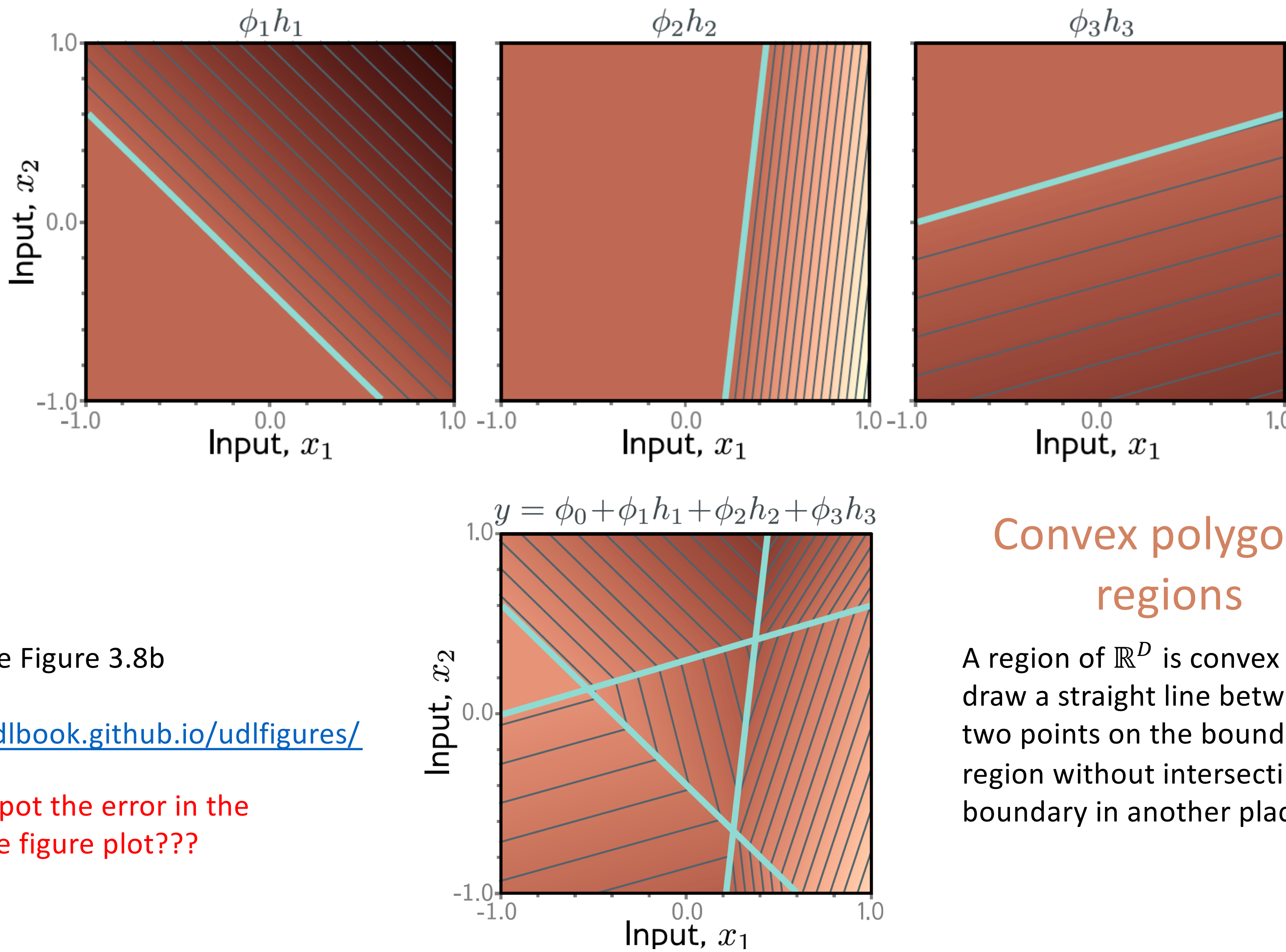


Weight the hidden units



Sum the weighted hidden units





Interactive Figure 3.8b

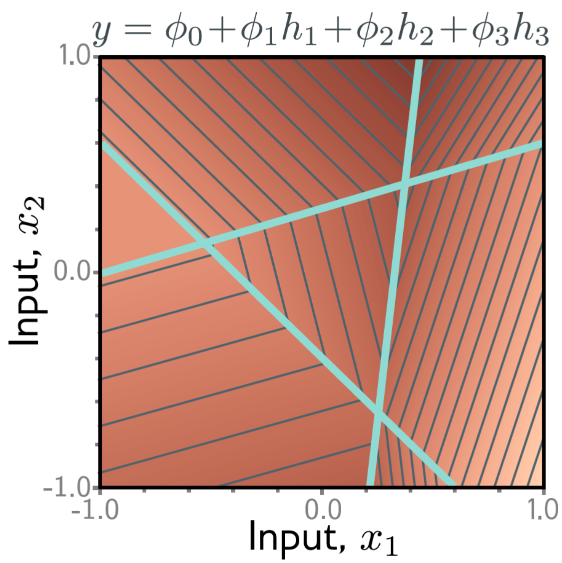
https://udlbook.github.io/udlfigures/

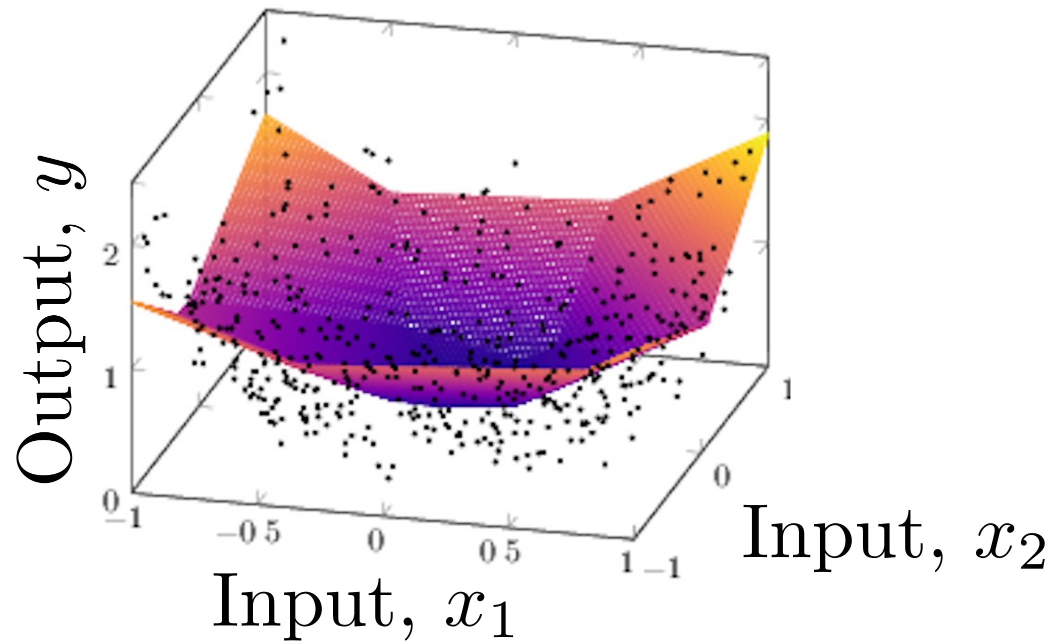
Can you spot the error in the interactive figure plot???

Convex polygonal regions

A region of $\mathbb{R}^D$ is convex if we can draw a straight line between any two points on the boundary of the region without intersecting the boundary in another place.

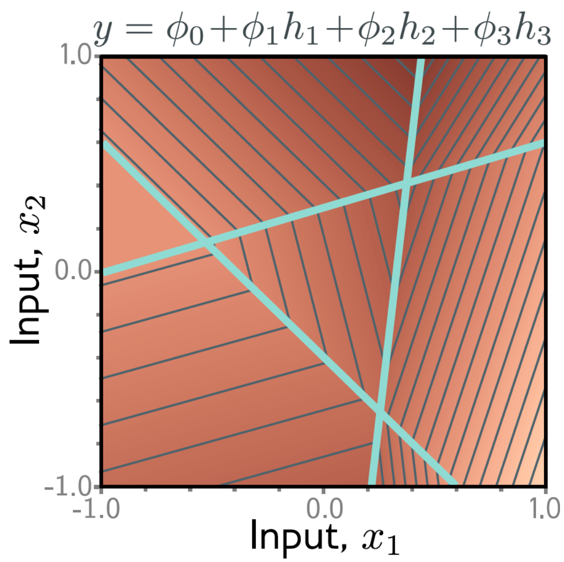
Fitting a dataset where:
each sample has 2 inputs and 1 output





Question:

- For the 2D case, what if there were two outputs?
- If this is one of the outputs, what would the other one look like?



(slido poll on next slide)

slido

Please download and install the Slido app on all computers you use

For the 2D case with 2 outputs, what would the other output look like?

ⓘ Start presenting to display the poll results on this slide.

Shallow neural networks

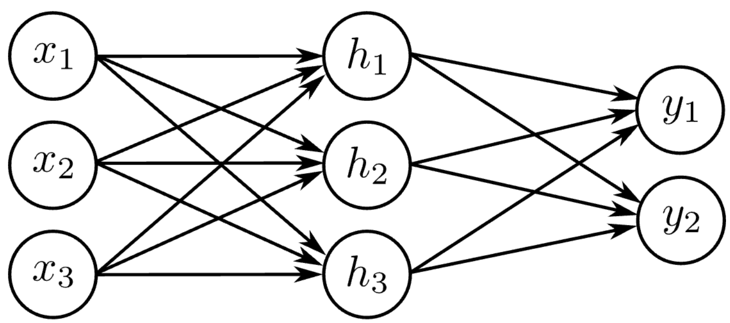
- Example network, 1 input, 1 ouput
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

Arbitrary inputs, hidden units, outputs

- D_i inputs, D hidden units, and D_o Outputs

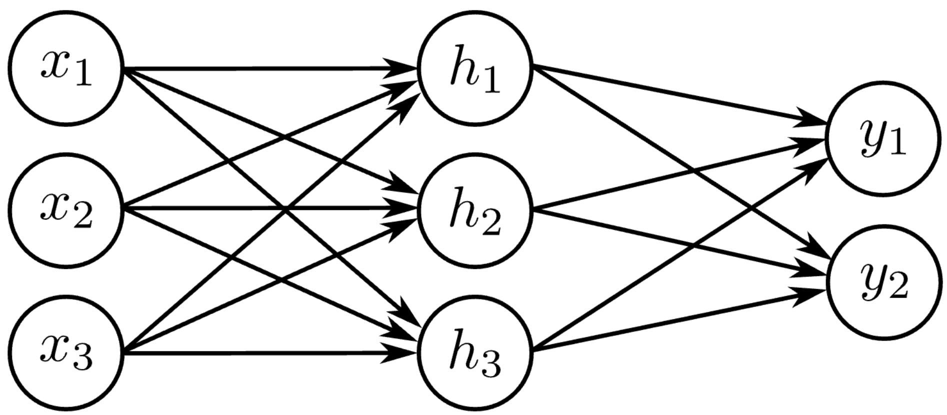
$$h_d = \mathrm{a}\left[\theta_{d0} + \sum_{i=1}^{D_i} \theta_{di} x_i\right] \qquad y_j = \phi_{j0} + \sum_{d=1}^{D} \phi_{jd} h_d$$

- e.g., Three inputs, three hidden units, two outputs



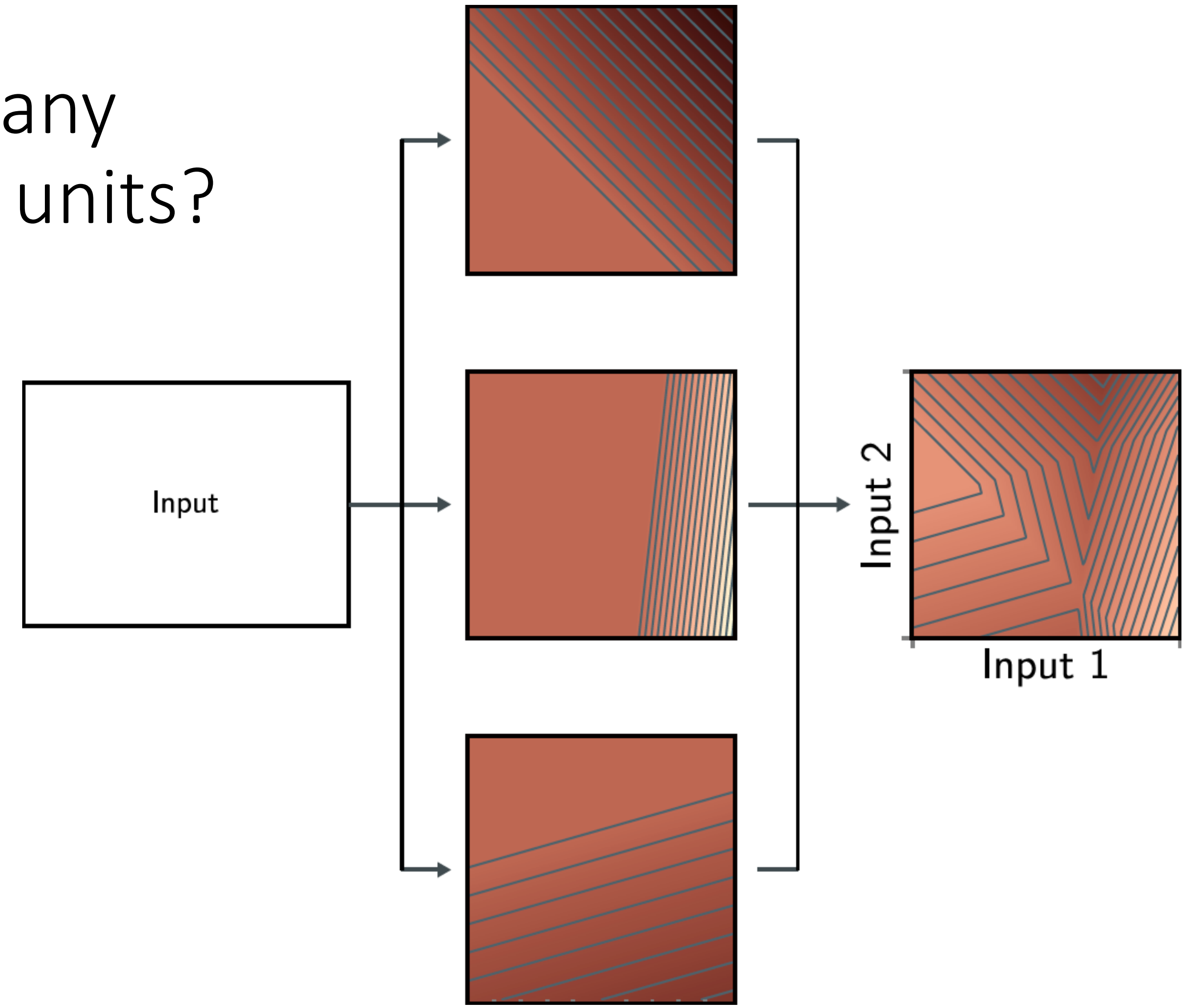
Question:

- How many parameters does this model have?

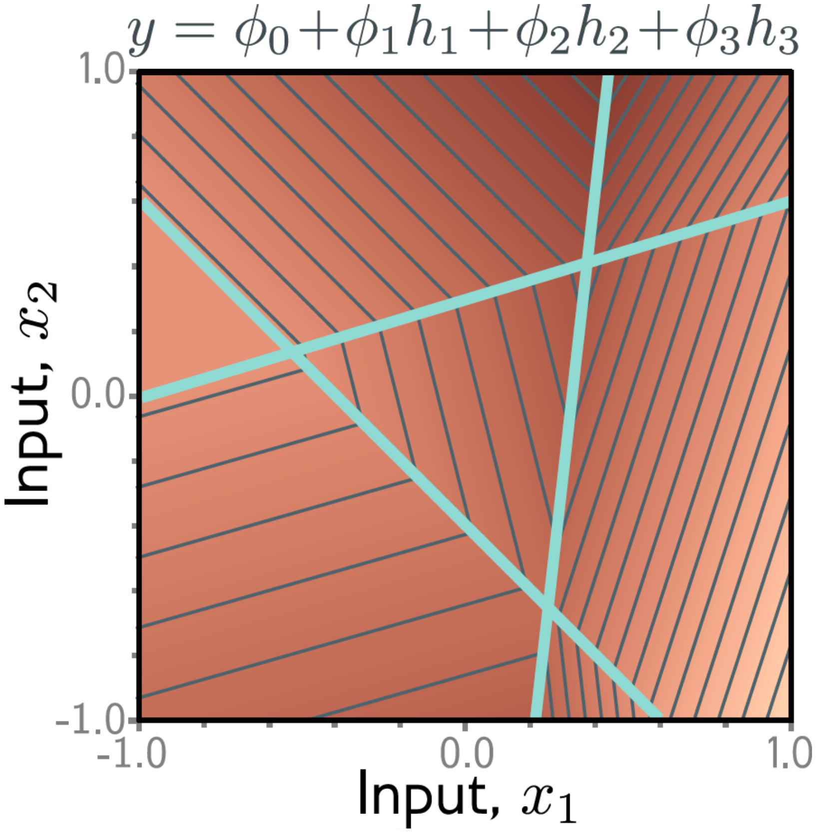


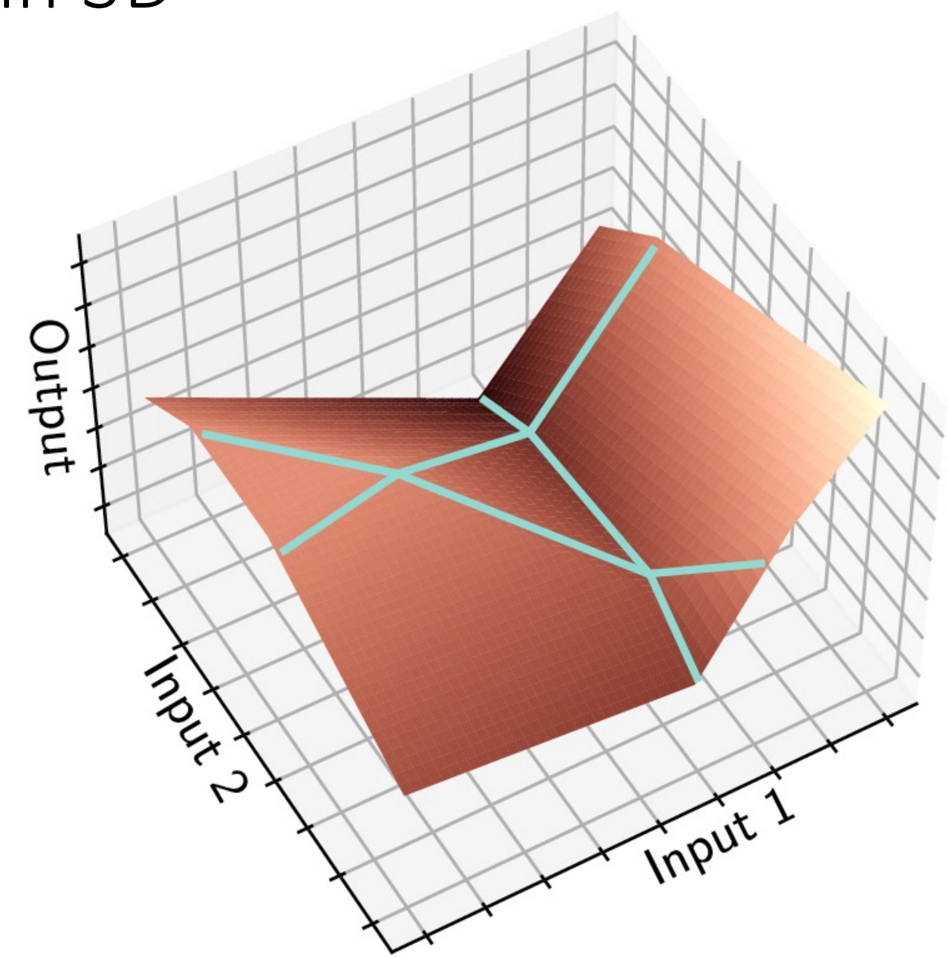
https://piazza.com/class/m5v834h9pcatx/post/19

How many hidden units?

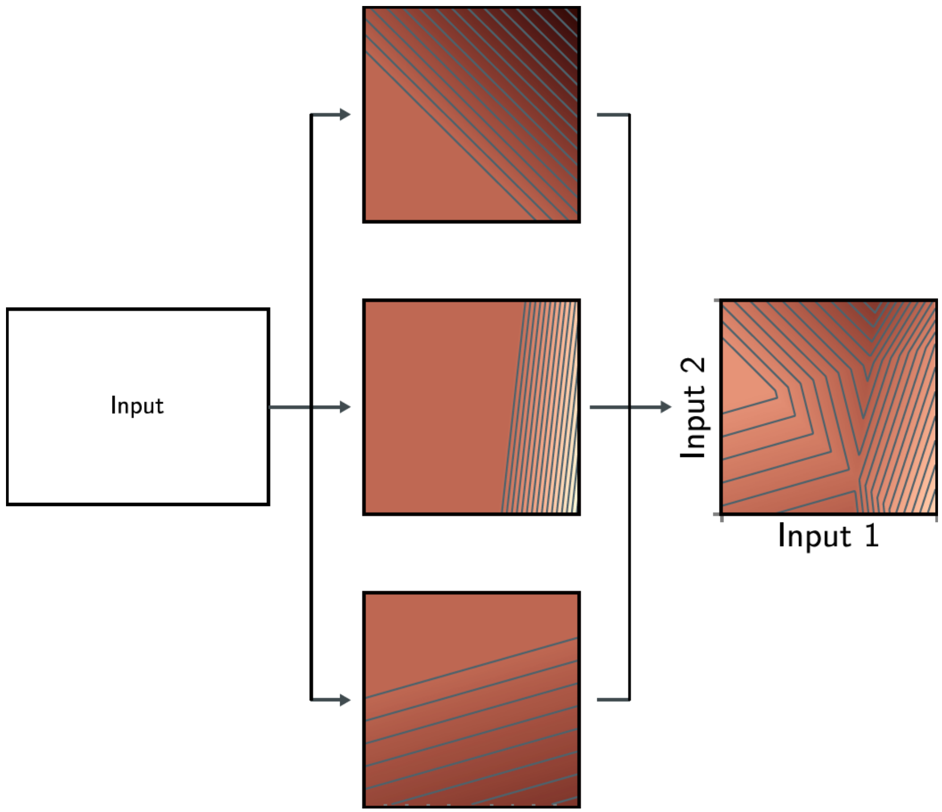


Output with boundaries and in 3D

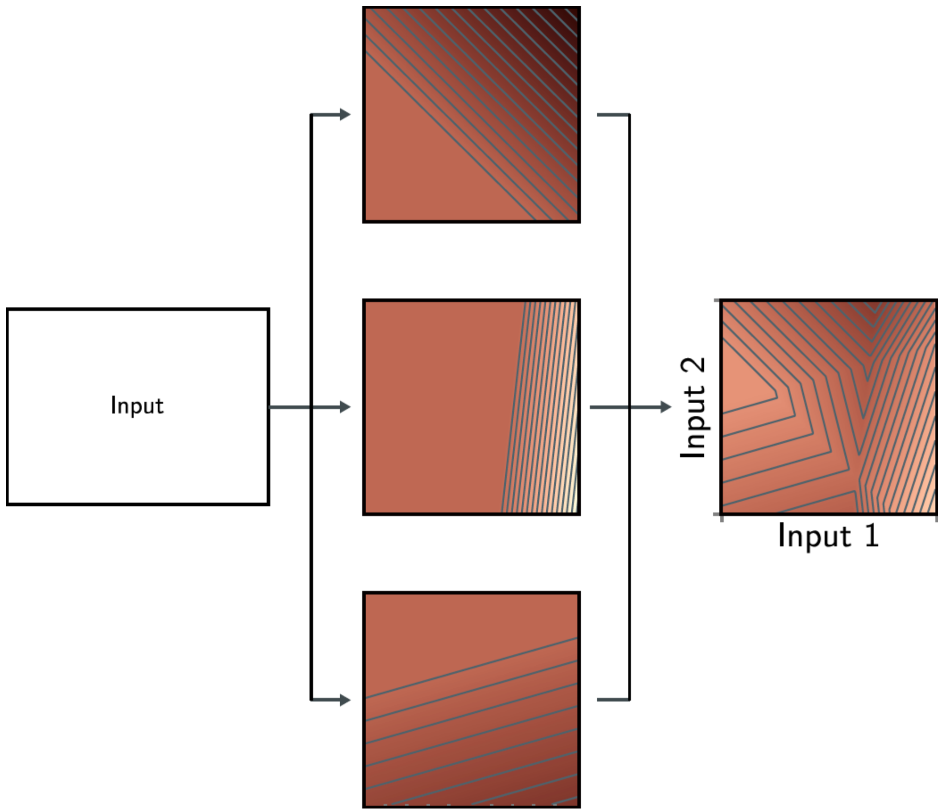


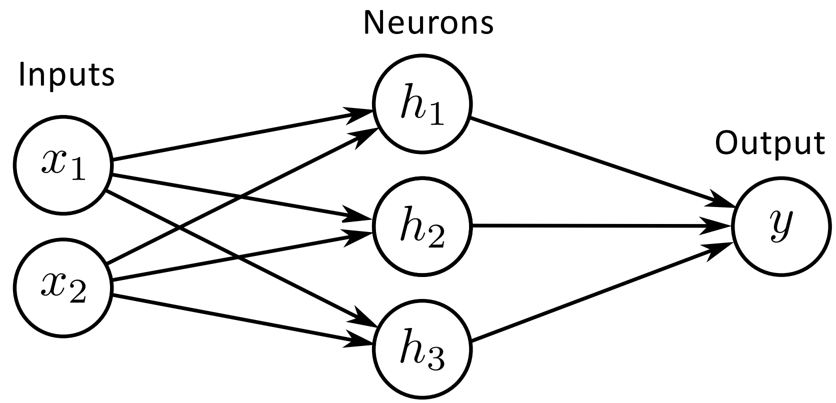


How would you draw and write this neural network?



How would you draw and write this neural network?





"neural network"

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

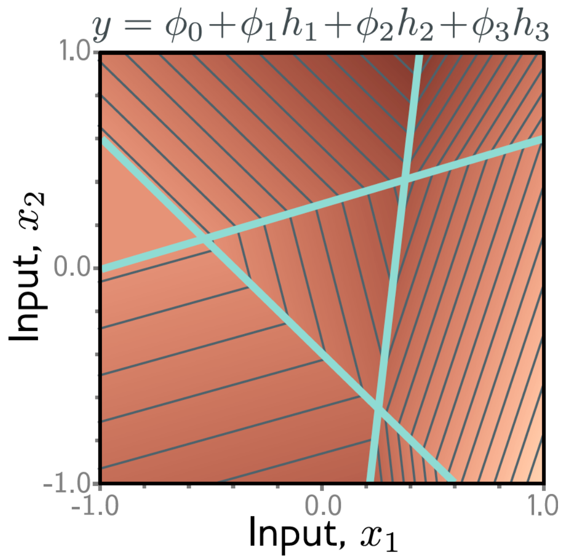
$$\begin{aligned} h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x_1 + \theta_{12}x_2] \\ h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x_1 + \theta_{22}x_2] \\ h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x_1 + \theta_{32}x_2] \end{aligned}$$

Shallow neural networks

- Example network, 1 input, 1 ouput
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

Number of output regions

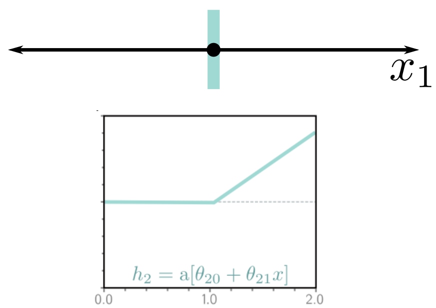
- With ReLU activations, each output consists of multi-dimensional piecewise linear hyperplanes
- With two inputs, and three hidden units, we saw there were seven polygons for each output:



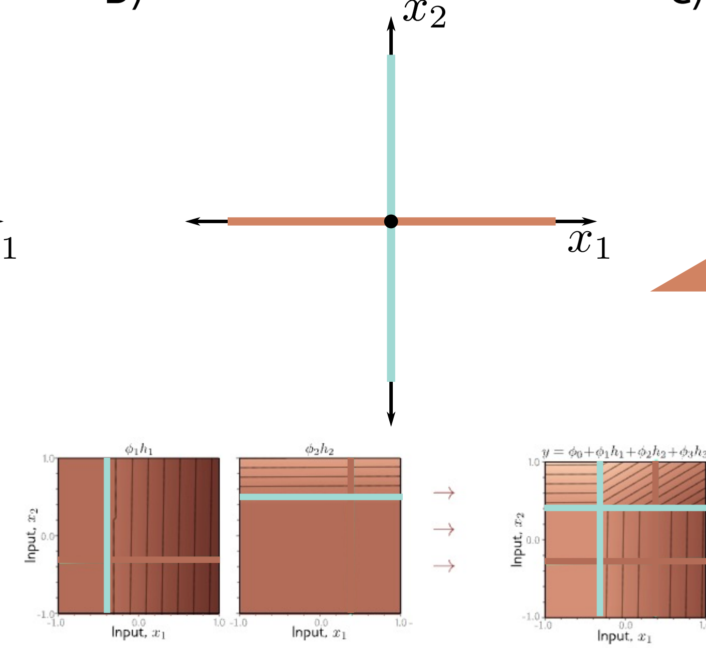
Example with $D = D_i \rightarrow 2^{D_i}$ regions

D_i : # of inputs
D : # of hidden units
D_o : # of outputs

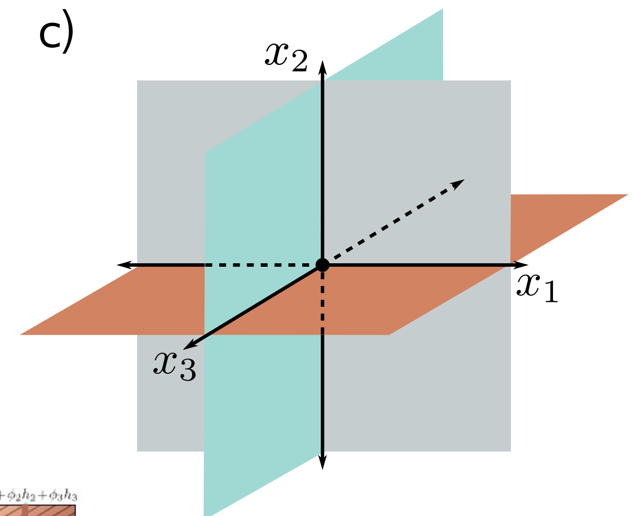
a)



b)



c)



- 1 input (1-dimension)
- 1 hidden unit
- creates two regions (one joint)

- 2 input (2-dimensions) with
- 2 hidden units
- creates four regions (two lines)

- 3 inputs (3-dimensions) with
- 3 hidden units
- creates eight regions (three planes)

Number of regions:

D_i : # of inputs
D : # of hidden units
D_o : # of outputs

- Number of regions created by $D > D_i$ hyper-planes in D_i dimensions was proved by Zaslavsky (1975) to be:

$$\sum_{j=0}^{D_i} \binom{D}{j} = \frac{D!}{j!(D-j)!}$$

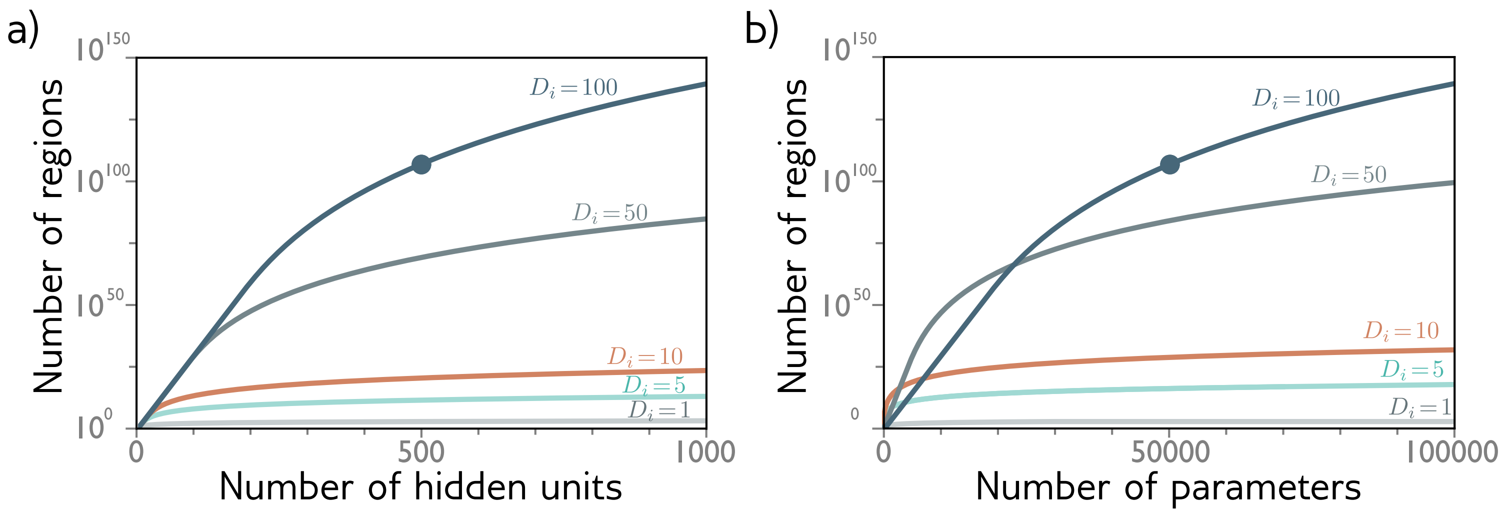
← Binomial coefficients!

- How big is this? It's greater than 2^{D_i} but less than 2^D.

Number of output regions

D_i : # of inputs
D : # of hidden units
D_o : # of outputs

- In general, each output consists of D dimensional convex polytopes
- How many?



Highlighted point = 500 hidden units or 51,001 parameters

Shallow neural networks

- Example network, 1 input, 1 ouput
- Universal approximation theorem
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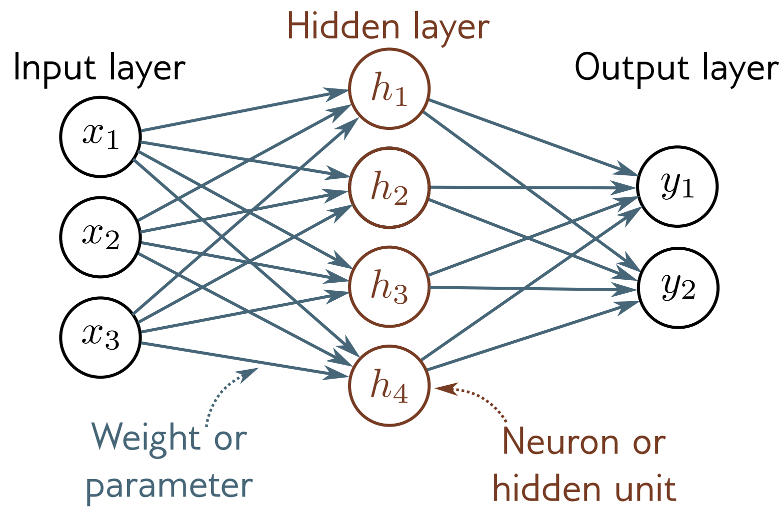
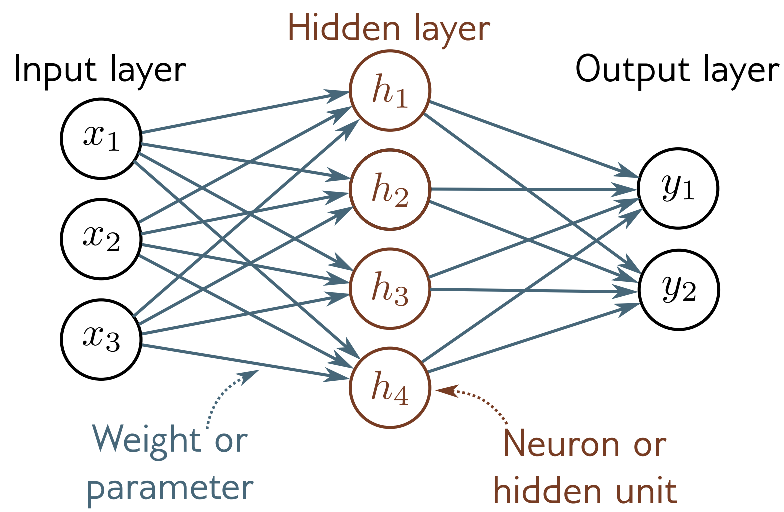
- Number of regions
- Terminology

Nomenclature

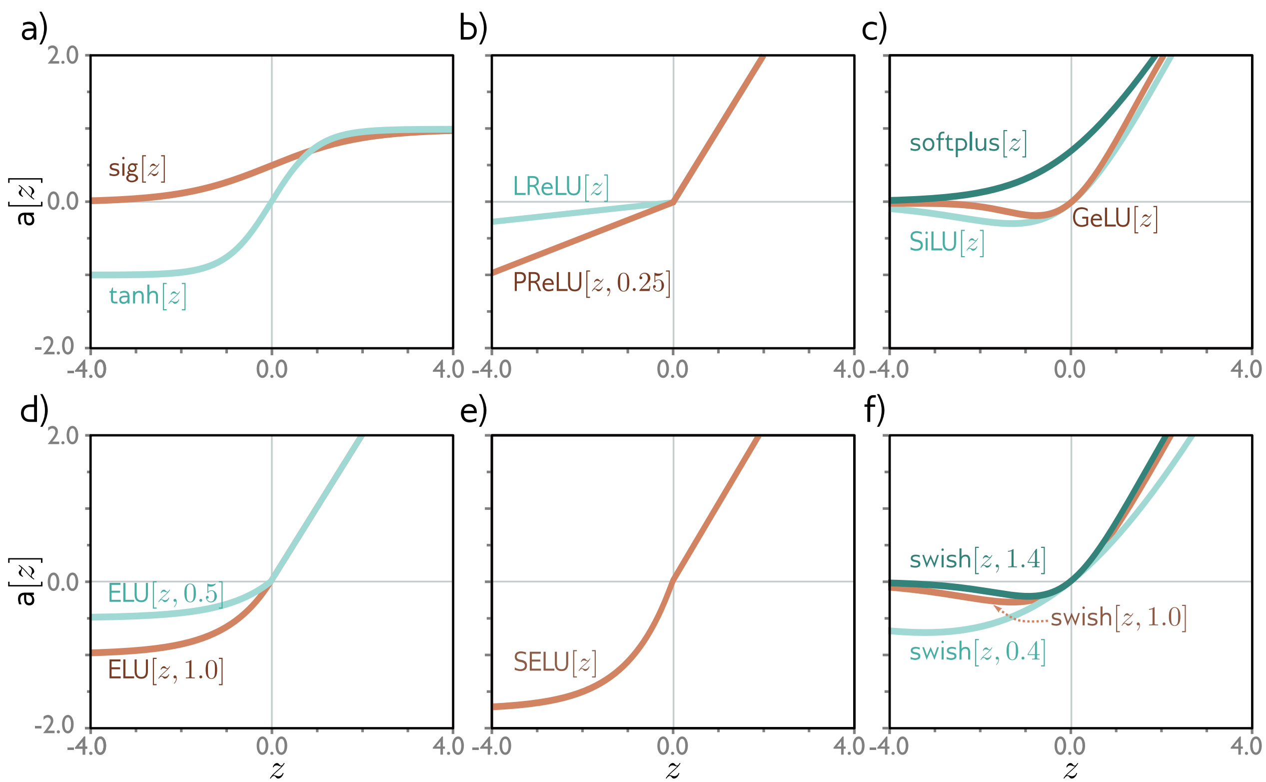


Nomenclature



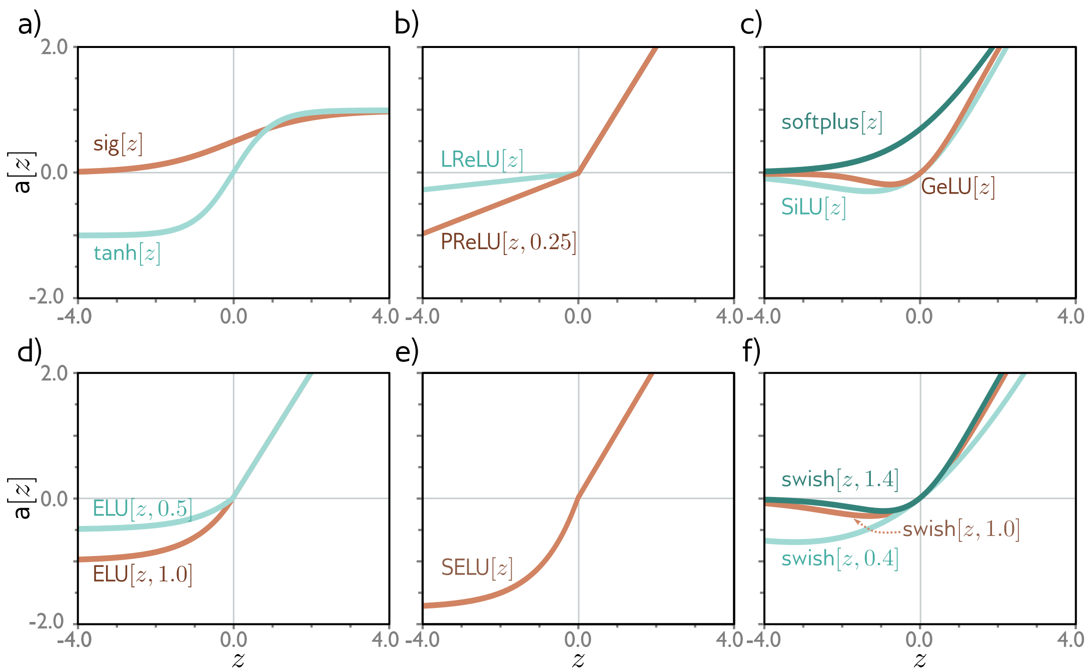
- Y-offsets = biases
- Slopes = weights
- Everything in one layer connected to everything in the next = fully connected network (multi-layer perceptron)
- No loops = feedforward network
- Values after ReLU (activation functions) = activations
- Values before ReLU = pre-activations
- One hidden layer = shallow neural network
- More than one hidden layer = deep neural network
- Number of hidden units $\approx$ capacity

Other activation functions



Ramachandran, P., Zoph, B., & Le, Q. V. (2017). Searching for activation functions. *arXiv:1710.05941*.

Interactive Figures 3.3b and 3.3c



Also look at

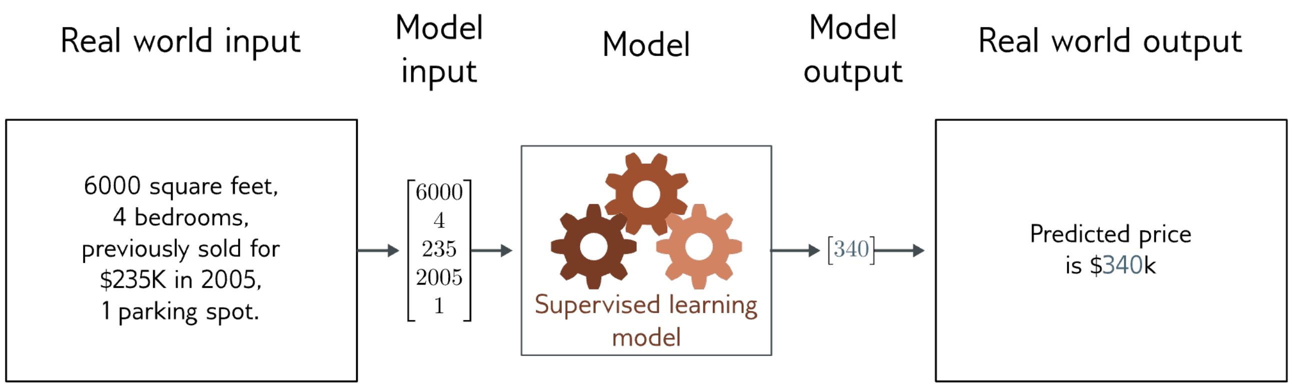
3.3b – 1D shallow network (sigmoid)

3.3c – 1D shallow network (Heaviside/Step)

$$\text{heaviside}[z] = \begin{cases} 0 & z < 0 \\ 1 & z \geq 0 \end{cases}$$

https://udlbook.github.io/udlfigures/

Regression



We have built a model that can:

- take an arbitrary number of inputs
- output an arbitrary number of outputs
- model a function of arbitrary complexity between the two

$$h_d = \mathrm{a}\left[\theta_{d0} + \sum_{i=1}^{D_i} \theta_{di} x_i\right] \qquad y_j = \phi_{j0} + \sum_{d=1}^{D} \phi_{jd} h_d$$

Next time:

- What happens if we feed one neural network into another neural network?

Feedback?

Link